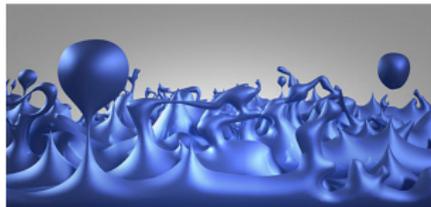


The conformal bootstrap

Silviu S. Pufu

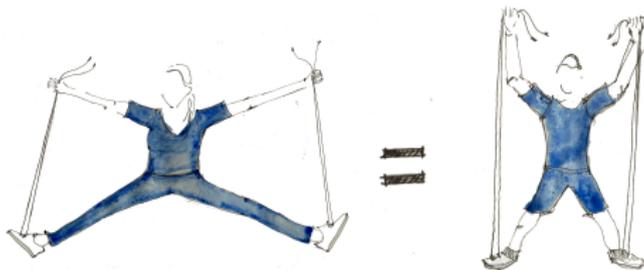
Princeton University

Seattle, July 23, 2022



The bootstrap philosophy

My task: Talk about bootstrap, with emphasis on bootstrapping string theory.



- **Bootstrap** = the use of **symmetry** and other principles (e.g. **unitarity/positivity, analyticity, crossing symmetry**) to constrain (or even determine) some physical quantity

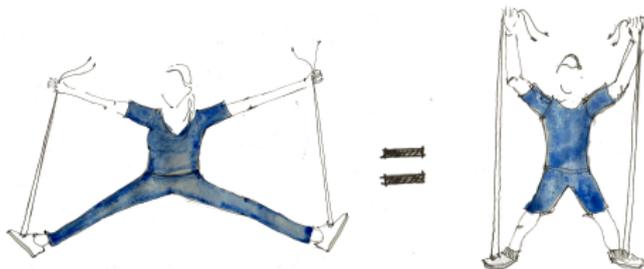
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- Has been applied to:

- scattering amplitudes
- matrix models
- lattice systems
- etc.

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- **This talk:** Focus on bootstrapping CFT.
- CFTs are QFTs with no intrinsic length/energy scale for which any angle-preserving transformation of spacetime points is a symmetry.
- May not be directly relevant for particle physics b/c CFTs have only massless excitations $E \propto |\vec{p}|$.
- **BUT:** CFTs are landmarks in the space of QFTs. They are endpoints of Renormalization Group (RG) flow, with RG flows connecting them



⇒ can deform CFTs to learn about more general QFTs [Hogervorst, Rychkov, van Rees, Katz, Fitzpatrick, Anand, Genest, Khandker, Walters, Xin, Chen, ...]

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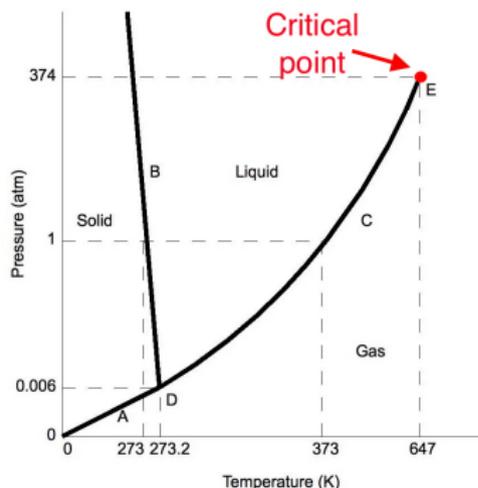


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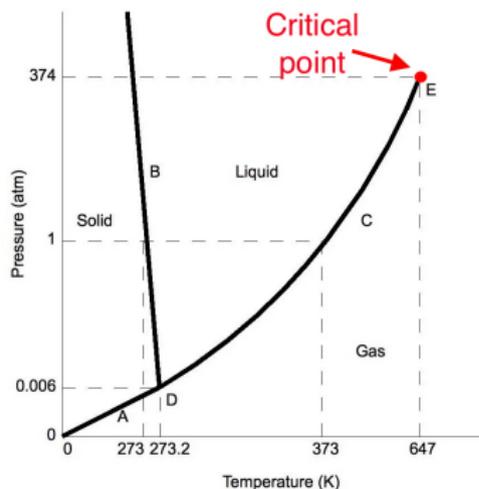


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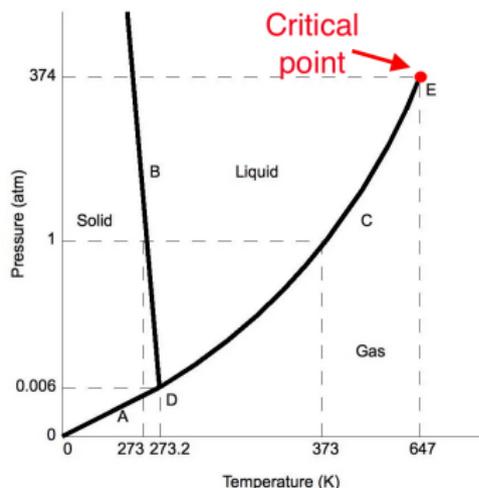


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CFT structure

- CFTs have **local operators** $\phi_i(\vec{x})$ and **correlation functions**
- Conformal symmetry \implies

$$\langle \phi_i(\vec{x}) \phi_j(\vec{y}) \rangle = \frac{\delta_{ij}}{|\vec{x} - \vec{y}|^{2\Delta_i}}$$

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where $\Delta_{ijk} \equiv \Delta_i + \Delta_j - \Delta_k$.

Higher pt correlators can be computed from these.

- In the context of phase transitions, the scaling dimensions Δ_i are related to “critical exponents”.
- For example,

$$\rho_\ell - \rho_v \propto (T_c - T)^\beta, \quad \beta = \frac{\Delta_\sigma}{3 - \Delta_\epsilon}$$

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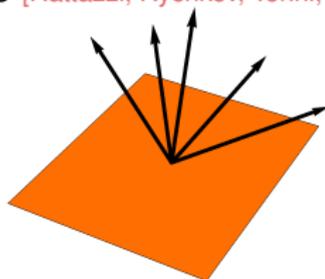
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Conformal bootstrap

- One can compute a 4-pt function $\langle \phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\phi_3(\vec{x}_3)\phi_4(\vec{x}_4) \rangle$ in two different ways, and they must agree:

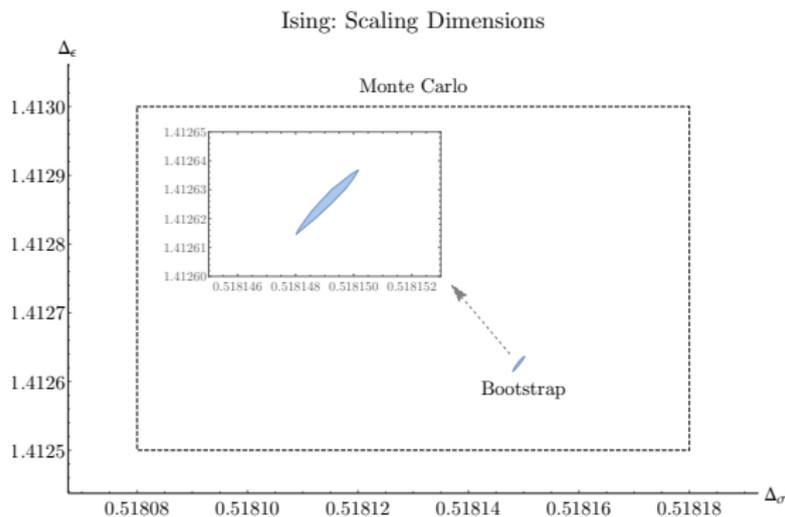
$$\langle \phi_1\phi_2\phi_3\phi_4 \rangle = \sum_k f_{12k} \phi_k \begin{array}{c} \phi_1 \qquad \phi_4 \\ \diagdown \quad \diagup \\ \text{---} \phi_k \text{---} \\ \diagup \quad \diagdown \\ \phi_2 \qquad \phi_3 \end{array} f_{34k} = \sum_k \begin{array}{c} \phi_1 \qquad \phi_4 \\ \diagdown \quad \diagup \\ \text{---} f_{14k} \text{---} \\ \diagup \quad \diagdown \\ \phi_2 \qquad \phi_3 \\ \text{---} f_{23k} \text{---} \\ \diagdown \quad \diagup \\ \phi_k \end{array}$$

- In general, unknown set of ϕ_k and unknown f_{ijk} .
- Looking for situations that make this impossible can be rephrased as a semi-definite programming problem (numerical optimization) \implies exclude regions in CFT data space [Rattazzi, Rychkov, Tonni, Vichi, El-Showk, Kos, Paulos, Poland, Simmons-Duffin, ...].



Numerical results

- “Precision islands”: 3d Ising, $O(N)$, Gross-Neveu-Yukawa models.
- Ising: IR fixed pt of $L = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \lambda\phi^4$ with m tuned to m_{cr} .

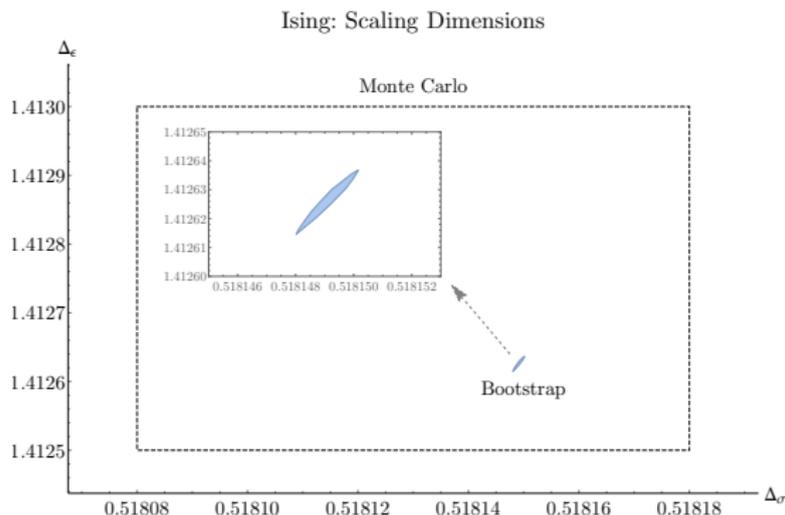


[Kos, Poland, Simmons-Duffin, Vichi '16]

- Most precise determination of critical exponents!

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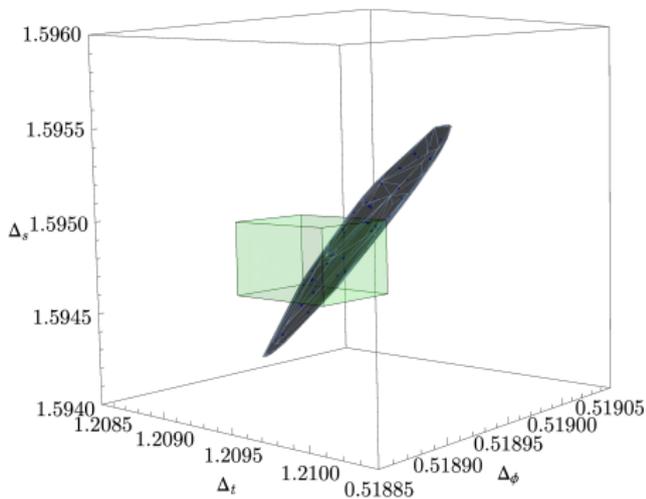
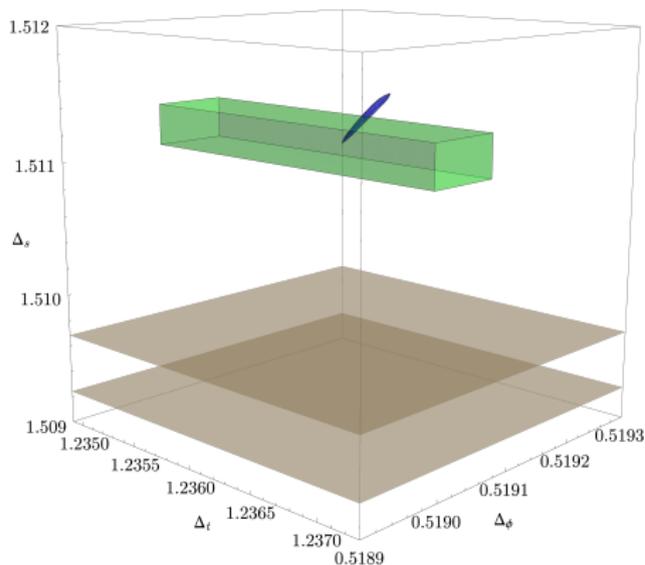
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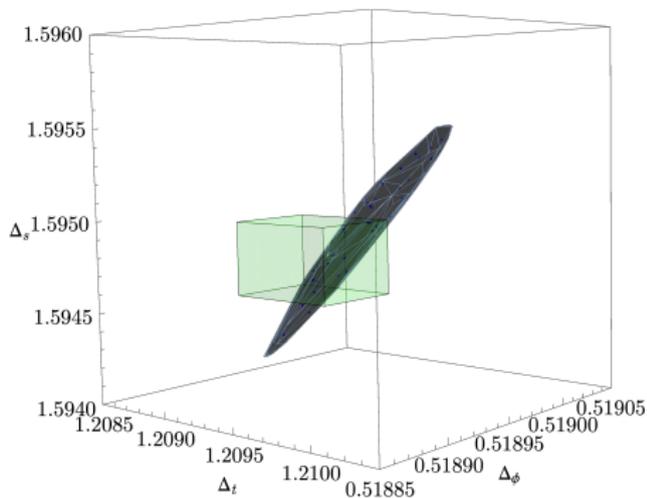
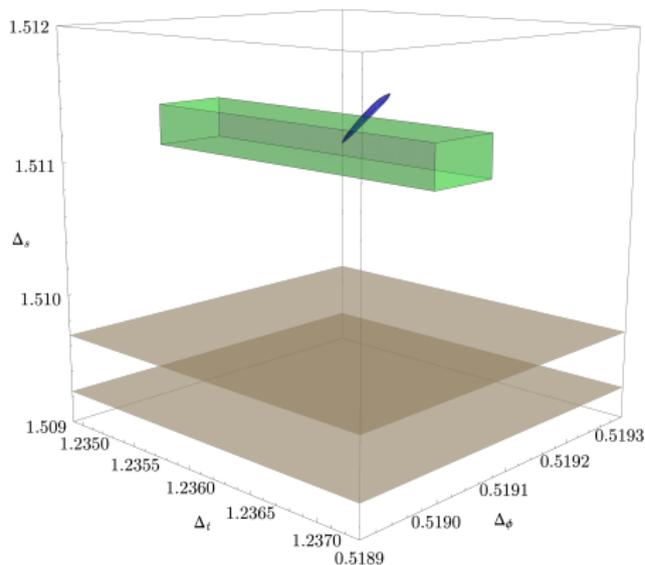
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[Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi '19, 20']

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Bootstrapping Quantum Gravity

- Rest of the talk: what bootstrap can teach us about Quantum Gravity and string theory.
- Connection: **AdS / CFT correspondence** (gauge / gravity duality; holographic duality) [Maldacena '97; Gubser, Klebanov, Polyakov '98; Witten '98] :
certain CFTs that are generalizations of QCD with N colors have dual descriptions in terms of gravity w/ negative cosmological constant (as a limit of string theory or M-theory)
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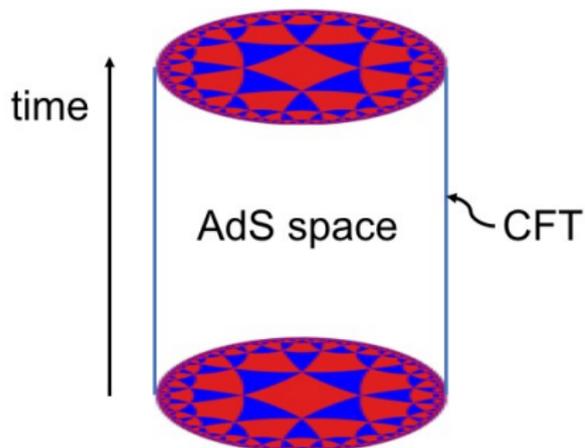
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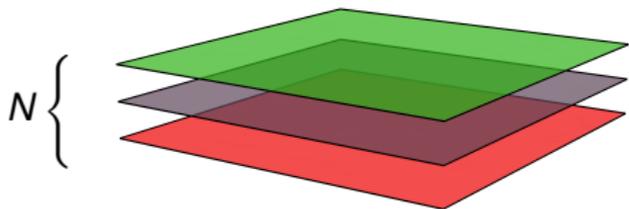
AdS/CFT

- Precise correspondence: CFT quantities (e.g. correlation functions) can be computed from the bulk.



- $L/\ell_p \propto N^\#$ where L is the curvature radius of AdS, and ℓ_p is the Planck length.
- (Semi-classical) gravity is a good approximation when the number of colors N is large ($\implies L/\ell_p$ is large, i.e. the curvature of spacetime is small)

Examples

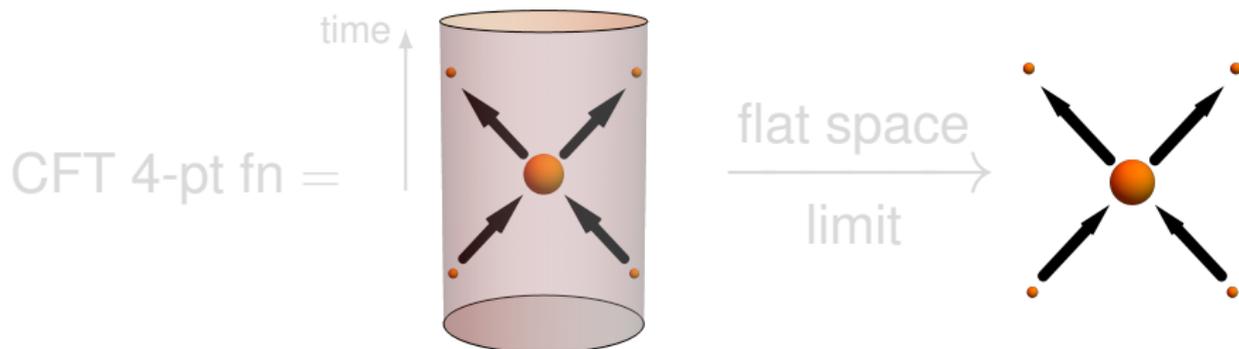


Examples of CFTs with AdS duals (both with maximal SUSY):

- $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in $3 + 1$ dimensions.
 - $L/\ell_p \propto N^{1/4}$.
 - Dual to $AdS_5 \times S^5$.
 - Describes physics of N membranes in 10-dimensional string theory.
- ABJM theory in $2 + 1$ dimensions [Aharony, Bergman, Jafferis, Maldacena '09].
 - $L/\ell_p \propto N^{1/6}$.
 - Dual to $AdS_4 \times S^7$.
 - Describes physics of N membranes in 11-dimensional M-theory.

Study Quantum Gravity through CFT

- What can one learn about string theory / Quantum Gravity by studying these CFTs?
- Example: **graviton scattering** [Polchinski, Giddings, Penedones, Fitzpatrick, Kaplan, Goncalves, ...]



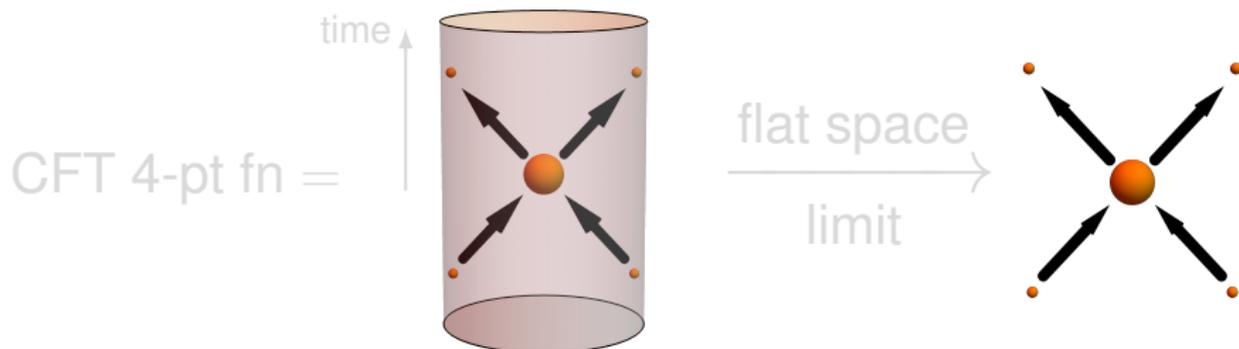
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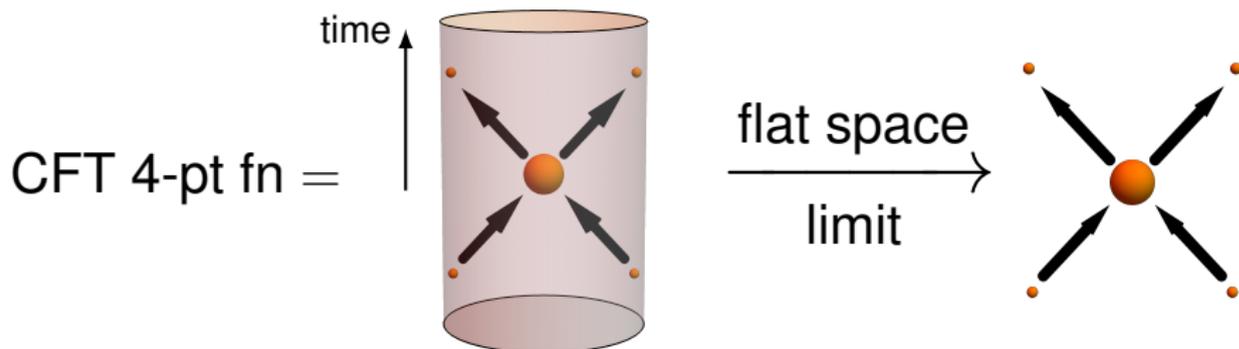
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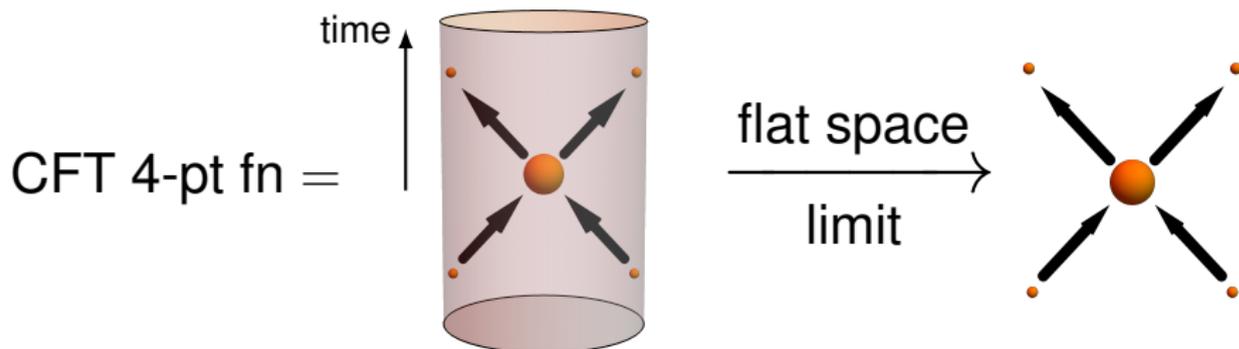
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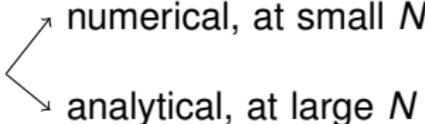
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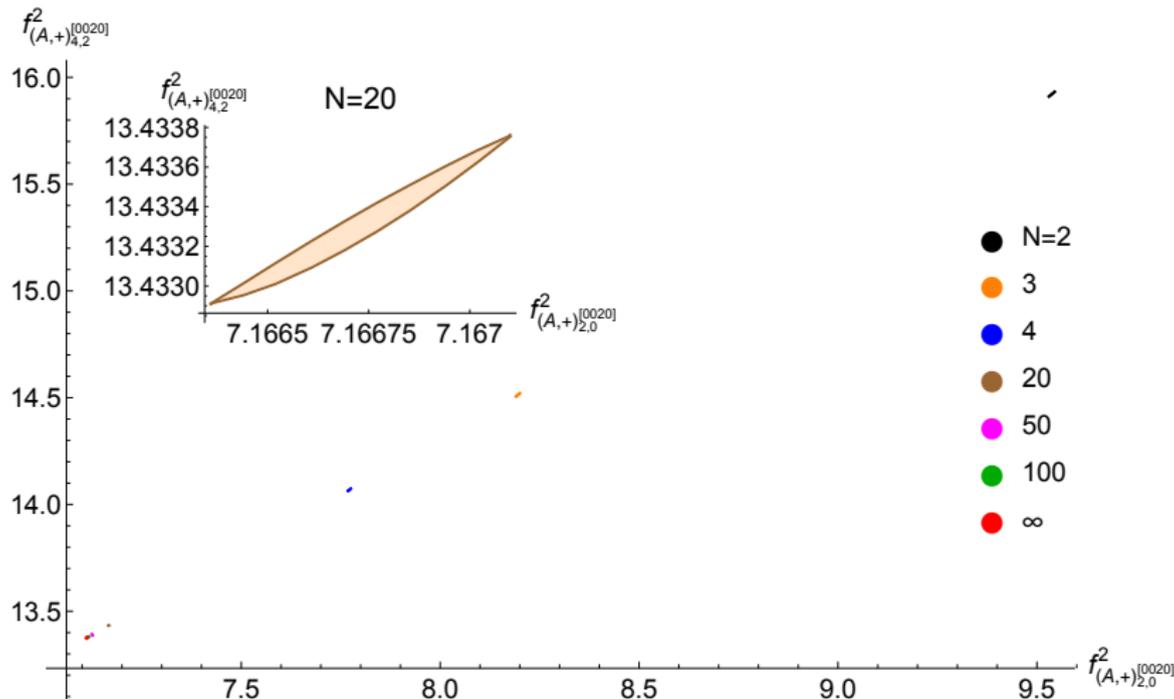
Of all CFTs w/ AdS duals, the most constrained are those with **supersymmetry** (such as $\mathcal{N} = 4$ SYM or ABJM theory), for which one can compute “**protected**” quantities.

- Example: integrals of correlators of certain operators.

Ongoing efforts 

Numerical study in ABJM theory (in $2 + 1$ dimensions)

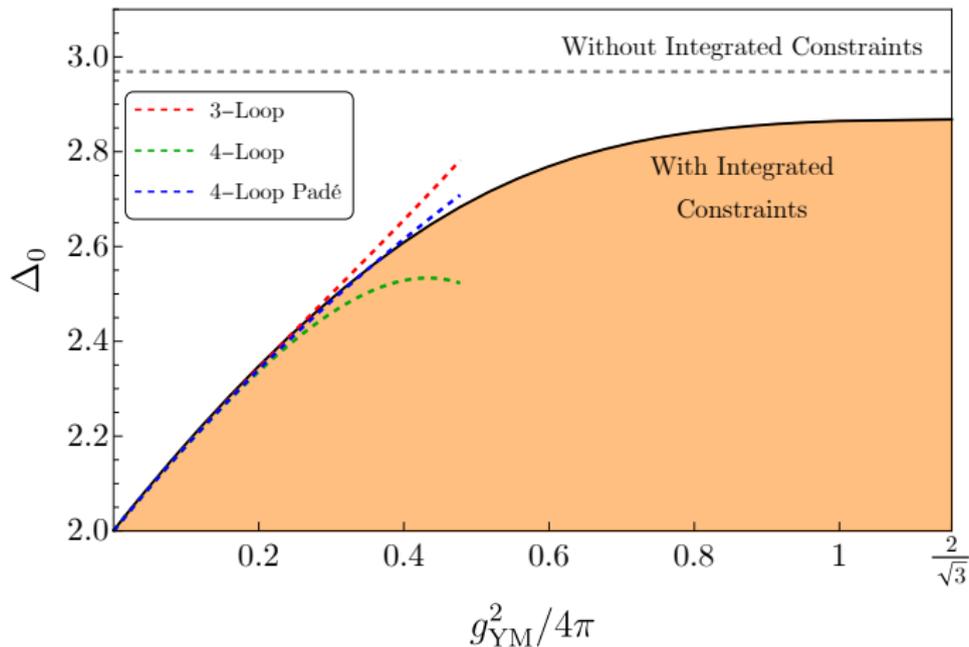
Numerical bootstrap **islands** for ABJM theory [Chester, Lee, SSP, Yacoby '14; Agmon, Chester, SSP '19] (uses SUSY [Dedushenko, Fan, SSP, Yacoby '16, '17, '18]).



Numerical study in $\mathcal{N} = 4$ SYM (in $3 + 1$ dimensions)

Numerical bootstrap **bounds** on lowest unprotected op in $SU(2)$ $\mathcal{N} = 4$ SYM

[Chester, Dempsey, SSP '19]



Bounds can be improved by including more SUSY info, mixed correlators, etc.

Analytic bootstrap: a 4d example

- In $\mathcal{N} = 4$ SYM, schematically at **fixed** g_{YM}

$$\langle TTTT \rangle = 1 + N^{-2} + N^{-\frac{7}{2}} + N^{-4} + N^{-\frac{9}{2}} + \dots$$

and each coefficient can be determined from Witten diagrams.

- Analytic properties of Witten diagrams + symmetry + exact results from SUSY completely determine $\langle TTTT \rangle$ at the first few orders in $1/N$.
- Flat space limit of $\langle TTTT \rangle \implies$ the first few corrections to the Einstein-Hilbert action (as seen by graviton scattering). Schematically:

$$S_{10d} = \int \sqrt{-g} \left[R + \text{Riemann}^4 + D^4 \text{Riemann}^4 + \dots \right]$$

[Chester, SSP, Yin '18; Binder, Chester, SSP, Wang '19; Binder, Chester, SSP '19; Chester, Green, SSP, Wang, Wen '19, '20] (See also [Alday, Bissi, Perlmutter, Heslop, Paul, ...]).

- Can derive the Riemann^4 , $D^4 \text{Riemann}^4$ terms in 10-dimensional string theory (and also in 11-dimensional M-theory from ABJM theory) **from CFT bootstrap + SUSY!**

Comments

- Bootstrap is **crucial**, b/c the standard technique (Witten diagrams in AdS) cannot be used due to the fact that the interaction vertices **are not fully known**.
- In $\mathcal{N} = 4$ SYM, $\langle TTTT \rangle$ is a fn of two params: N and g_{YM} .
 - $g_{\text{YM}}^2 \leftrightarrow g_s$ (string coupling)
 - $1/N \rightarrow$ (small momentum expansion of scattering amplitudes)
- In flat space, graviton S-matrix is known:
 - At leading order in g_{YM} exactly in momentum (Virasoro-Shapiro)
 - At low orders in momentum ($R^4, D^4 R^4, D^6 R^4$) exactly in g_{YM} .
- Two concrete goals:
 - Determine AdS analog of Virasoro-Shapiro
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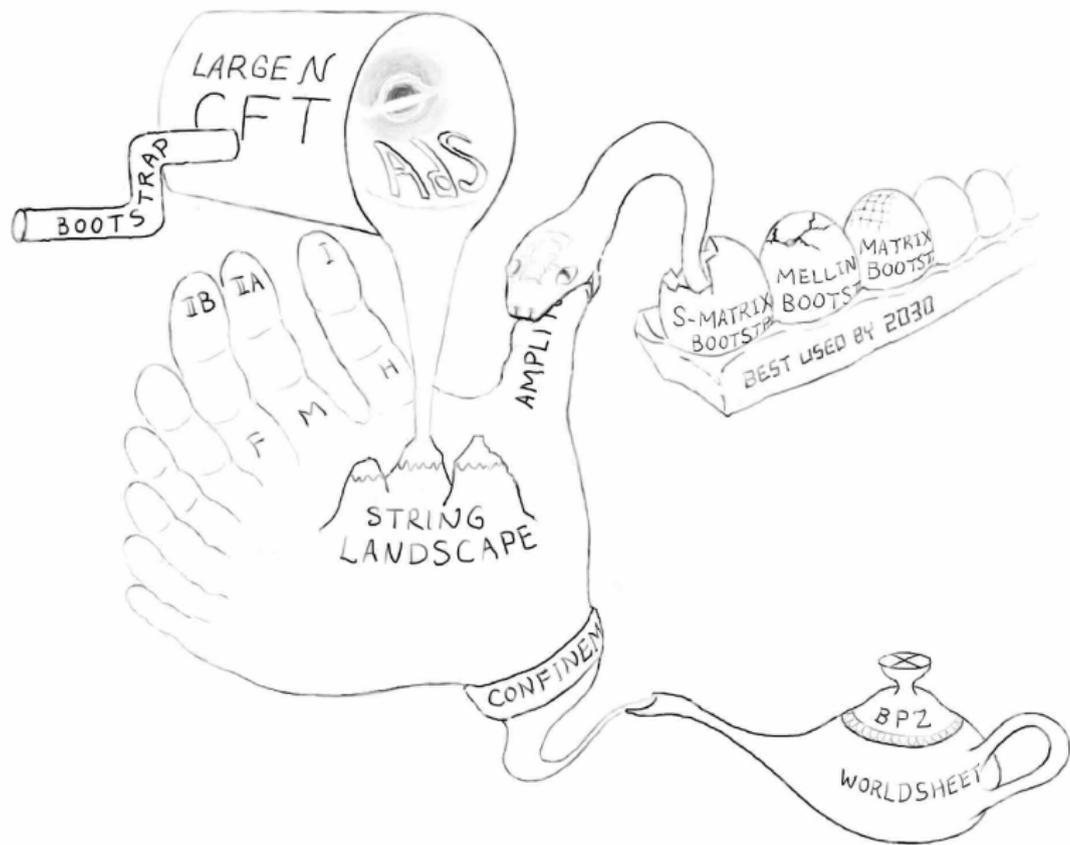
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Conclusion and other directions

- CFT gives a universal language that applies to a variety of systems that exhibit 2nd order phase transitions and even to **Quantum Gravity**.

Other ways to bootstrap string theory:

- Bootstrap S-matrices directly. Lots of progress recently. [Vieira, Penedones, Guerrieri, Caron-Huot, Rastelli, Simmons-Duffin, Mazac, ...]
- Bootstrap worldsheet CFT (perhaps together with dual CFT).
- Instead of OPE decomposition, use a different expansion of the correlation function. (Mellin bootstrap?) [Gopakumar, Kaviraj, Sen, Sinha, ...]
- Finite temperature bootstrap [Iliesiu, Kologlu, Mahajan, Perlmutter, Simmons-Duffin, ...]
- Swampland questions: is it possible to rule in/out scale separated AdS vacua? String universality?
- Bootstrapping de Sitter Quantum Gravity [Arkani-Hamed, Baumann, Lee, Pimentel, Pajer, Stefanyszyn, Supel, Goodhew, Jazayeri, Sleight, Taronna, Duaso Pueyo, Joyce, Meltzer, Di Pietro, Gorbenko, Komatsu, Hogervorst, penedones, Vaziri, ...]



[Illustration by Xi Yin, from [\[Gopakumar, Perlmutter, Pufu, Yin '22\]](#)]